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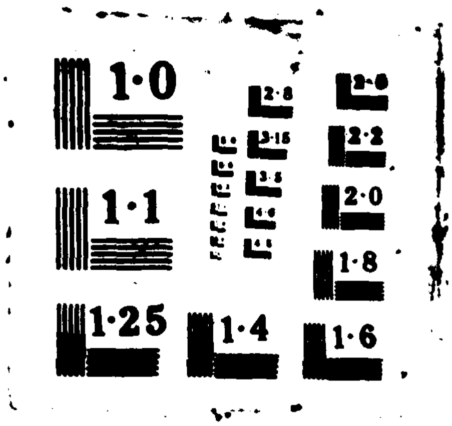
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ROBUSTNESS OF t-TEST

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ABSTRACT

In this paper we establish the optimality robustness of Student's t-test mainly without invariance. This generalizes some well-known results of Lehmann and Stein (1949), and Kariya and Eaton (1977).

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1. INTRODUCTION

The standard optimum properties like UMP, UMPU, UMPS, UMPI, of the familiar Student's t-test for testing the significance of the mean of a univariate normal population with unknown variance against one-sided or both-sided alternatives are well-known. See, for example, Lehmann (1959). In this paper we show mainly without invariance that the t-test continues to be optimum when normality is replaced by a class of elliptically symmetric distributions, thereby establishing optimality robustness of the t-test. To fix ideas, let $\underset{\sim}{x} = (x_1, \ldots, x_n)' \sim N(\mu \underset{\sim}{1}, \sigma^2 I_n)$, $\sigma^2 > 0$ unknown, and consider the problem of testing $H_0^*: \mu = 0$ versus $H_1^*: \mu > 0$ or $H_2^*: \mu \neq 0$. Define the one-sided t-test and two-sided t-test respectively by

$$\phi_1(\underset{\sim}{x}) = \begin{cases} 1 \text{ if } v > c \\ 0 \text{ otherwise} \end{cases} \tag{1.1}$$

$$\phi_2(\underset{\sim}{x}) = \begin{cases} 1 \text{ if } |v| > d \\ 0 \text{ otherwise} \end{cases} \tag{1.2}$$

where $v = \underset{\sim}{x}'\underset{\sim}{1}/\|\underset{\sim}{x}\| \; \|\underset{\sim}{1}\|$, $\|\underset{\sim}{x}\| = (\sum_1^n x_i^2)^{1/2}$, $\|\underset{\sim}{1}\| = n^{1/2}$. It is proved in Lehmann and Stein (1948) that

(a) for $\frac{1}{2} \leq \alpha < 1$, ϕ_1 is UMP for H_0^* versus H_1^*; however,

(b) no UMP test exists for H_0^* versus H_1^* for $0 < \alpha < \frac{1}{2}$. On the other hand, it is trivial that

(c) ϕ_1 is UMPI for H_0^* versus H_1^* while

(d) ϕ_2 is both UMPS and UMPI for H_0^* versus H_2^*.

In the sequel, we assume that $\underset{\sim}{x}$ has a density of the form

$$f(\underset{\sim}{x}|\mu,\sigma^2) = \sigma^{-n} q(\|\underset{\sim}{x} - \mu\underset{\sim}{1}\|^2/\sigma^2) \tag{1.3}$$

for some $q \in Q$ given by

$$Q = \{q \mid q\colon [0,\infty) \to [0,\infty),\ \int_{R^n} q(\|\underset{\sim}{x}\|^2)d\underset{\sim}{x} = 1\}. \tag{1.4}$$

In what follows we mostly assume that both σ^2 and q in (1.3) are unknown, and consequently define the three classes of densities on R^n:

$$\begin{aligned} F_0(\mu) &= \{f \mid f(\underset{\sim}{x}) = q(\|\underset{\sim}{x} - \mu\underset{\sim}{1}\|^2),\ q \in Q\} \qquad (1.5)\\ F_1(\mu) &= \{f \mid f \in F_0(\mu),\ \ q \textit{ nonincreasing}\}\\ F_2(\mu) &= \{f \mid f \in F_1(\mu),\ \ q \text{ convex}\}. \end{aligned}$$

The restrictions of nonincreasingness and convexity imposed on q in (1.5) above are standard in robustness study (see Kariya (1981)). Clearly $F_2(\mu) \subset F_1(\mu) \subset F_0(\mu)$ for all real μ. Denote the density of $\underset{\sim}{x}$ by $h(\underset{\sim}{x})$ and consider the testing problems

$$\begin{aligned} &H_0\colon h \in F_0(0) \text{ versus } K_0\colon h \in F_0(\mu),\ \ \mu > 0 \text{ unknown} \qquad (1.6)\\ &H_1\colon h \in F_1(0) \text{ versus } K_1\colon h \in F_1(\mu),\ \ \mu > 0 \text{ unknown}\\ &H_2\colon h \in F_2(0) \text{ versus } K_2\colon h \in F_2(\mu),\ \ \mu \neq 0 \text{ unknown}. \end{aligned}$$

Note the very general nature of both null and alternative hypotheses in (1.6) which are essentially nonparametric. We define by C_i^α the class of level α tests under H_i, $i = 0, 1, 2$, defined as

$$C_i^\alpha = \{\phi \in C^\alpha \mid E_f\phi \le \alpha \text{ for all } f \in F_i(0)\},\ \ 0 < \alpha < 1, \tag{1.7}$$

where C^α denotes the class of all level α tests on R^n. It then follows that $C_0^\alpha \subset C_1^\alpha \subset C_2^\alpha$ become $F_2(0) \subset F_1(0) \subset F_0(0)$.

There are some results in the literature for the testing problems similar to (1.6). Lehmann and Stein (1949) proved that (e)ϕ_1 is UMP for testing H_0 versus h is $N(\mu \underset{\sim}{1}, \sigma^2 I_n)$, $\mu > 0$ unknown. Kariya and Eaton (1977) proved that (f)ϕ_1 is UMP for testing H_0 versus K_1 while (g)ϕ_2 is UMPS for testing H_0 versus K_2. It is interesting to note the implications of these results. Clearly (f) is stronger than (e). Moreover, the class of level α tests for which ϕ_1 is UMP in (e) is precisely C_0^α and not $C_N^\alpha = \bigcap_{\sigma^2 > 0} C_N^\alpha(\sigma^2)$ which is relevant in (b). Here $C_N^\alpha(\sigma^2) = \{\phi \in \mathcal{C}^\alpha | E[\phi | N(0, \sigma^2 I_n)] \leq \alpha\}$, and C_N^α is clearly much bigger than C_0^α. The same is true for the optimality properties in (f) and (g) in that they hold only in C_0^α. For example, the UMP property of ϕ_1 in (f) simply means that for any $f(\cdot|\mu) \in F_1(\mu)$ and $\mu > 0$,

$$E[\phi_1 | f(\cdot|\mu)] \geq E[\phi | f(\cdot|\mu)] \quad \text{for all } \phi \in C_0^\alpha \tag{1.8}$$

but it does not mean that for a fixed $h(\cdot|\mu) \in F_1(\mu)$, ϕ_1 is UMP for testing

$$H_1(h): \mu = 0 \text{ versus } K_1(h): \mu > 0 \tag{1.9}$$

in the class of level α tests under the fixed h defined as

$$\mathcal{C}^\alpha(h) = \{\phi \in \mathcal{C}^\alpha | E[\phi | h(\cdot|0)] \leq \alpha\}. \tag{1.10}$$

It is, therefore, evident that in the consideration of optimality robustness, it is essential to clearly specify the class in which optimality holds.

It is the object of this paper to strengthen the results in (f) and (g). We prove in Section 2 that for each fixed $h \in F_1(\mu)$, ϕ_1 is UMP for testing $\mu = 0$ versus $\mu > 0$, i.e. for the problem (1.9), in the class of conditional level α tests given $w = \|\underset{\sim}{x}\|^2$, which is a subclass of $\mathcal{C}^\alpha(h)$ given in (1.10) above. Also, the class C_0^α is shown to be the intersection of the classes of conditional level α tests over $h \in F_1(0)$. These two results imply (f). Further, when $\frac{1}{2} \leq \alpha < 1$, ϕ_1 is shown to be UMP in $\mathcal{C}_\alpha = \bigcap_{h \in F_1(0)} \mathcal{C}^\alpha(h)$. In Section 3, it is shown that, for each fixed $h \in F_2(\mu)$, ϕ_2 is UMP for testing $\mu = 0$ versus $\mu \neq 0$ in the class of conditional similar level α tests satisfying a certain condition. This result is stronger than (g). Finally, in Section 4, some properties of the t-test with invariance are pointed out.

The following result, which is very basic and whose proof is omitted, is useful for our purpose.

<u>PROPOSITION 1</u>. Let $h \in F_0(\mu)$ be the density of $\underset{\sim}{x}$, where $h(\underset{\sim}{x}) = q(\|\underset{\sim}{x} - \mu\underset{\sim}{1}\|^2)$. Then

(1) (v,w) is a sufficient statistic for h.

(2) The joint density of (v,w) is given by

$$g(v,w;\,\mu) = c_0 q(w - 2\sqrt{w}\,v\,\sqrt{n}\,\mu + n\mu^2) r_0(v) w^{n/2-1}$$

with

$$r_0(v) = 2[B(\tfrac{1}{2}, \tfrac{n-3}{2})]^{-1}(1 - v^2)^{(n-3)/2}$$

where $c_0 = \{\Gamma(\frac{1}{2})\}^n/\Gamma(n/2)$ and $B(a,b)$ denotes the Beta function.

(3) When $\mu = 0$, v and w are independent with density $r_0(v)r_1(w)$ where

$$r_1(w) = c_0 w^{n/2-1} q(w).$$

By (1) of the above proposition, without loss of generality we can restrict the class of tests for problems (1.6) to the one based on (v.w). We denote this class by $\mathcal{D}$ in the rest of the paper.

2. ONE-SIDED TESTING PROBLEMS

As mentioned above, in this section we consider the following one-sided testing problems:

$$\text{For a fixed } h \in F_1(\mu), \quad \mu = 0 \text{ versus } \mu > 0 \tag{2.1}$$

$$H_0: h \in F_0(0) \text{ versus } K_1: h \in F_1(\mu), \quad \mu > 0 \text{ unknown} \tag{2.2}$$

$$H_1: h \in F_1(0) \text{ versus } K_1: h \in F_1(\mu), \quad \mu > 0 \text{ unknown.} \tag{2.3}$$

Note that the classes of level α tests relevant for these problems are respectively $C^\alpha(h)$, C_0^α and C_1^α defined in (1.10) and (1.7). Of course, by Proposition 1, any test function $\phi(\underset{\sim}{x})$ belonging to these classes is essentially of the form $\phi(v,w)$. Let $E_0^v(\cdot)$ denote the expectation of $\cdot$ with respect to the density $r_0(v)$ of v given in (2) of Proposition 1 and let a.a.(w,h) denote "almost all w with respect to the density $r_1(w)$ of w" given in (4) where r_1 depends on h. Also, let

$$\begin{aligned} \mathcal{D}^\alpha(h) &= \{\phi \in \mathcal{D}^\alpha \mid \phi(v,w) \in C^\alpha(h)\}, \quad h \in F_1(0) \\ \mathcal{D}_i^\alpha &= \{\phi \in \mathcal{D}^\alpha \mid \phi(v,w) \in C_i^\alpha\}, \quad i = 0, 1 \end{aligned} \tag{2.4}$$

where $\mathcal{D}^\alpha$ denotes the class of level α tests based on (v,w).

Our first main result is the following.

<u>THEOREM 2.1</u>. For problem (2.1), the test ϕ_1 defined in (1.1) is UMP in the class of conditional level α tests

$$K^\alpha(h) = \{\phi \in \mathcal{D}^\alpha(h) \mid E_0^v[\phi(v,w)] \le \alpha \quad \text{a.a.}(w,h)\}. \tag{2.5}$$

Proof. Write $h(\underset{\sim}{x}) = q(\|\underset{\sim}{x} - \mu \underset{\sim}{1}\|^2)$ in $F_1(\mu)$ and fix $\mu = \mu_1 > 0$. From (2) of Proposition 1, the conditional density of v given w is given by $g(v,w;\ \mu_1)/\int_{-1}^{1} g(v,w;\ \mu_1)dv$. Using the nonincreasing property of q and and applying the Neyman-Pearson Lemma, it follows that the test with the critical region $v > c(w)$ is MP in the class (2.5). But, when $\mu = 0$, v and w are independent from (3) of Proposition 1. Therefore, c(w) is independent of w. Finally, $r_0(v)$ being independent of q, the above test coincides with ϕ_1 and is UMP in (2.5), thus completing the proof.

Remark 2.1. It is important to note that ϕ_1 is UMP only in $K^\alpha(h)$ but not in $\mathcal{D}^\alpha(h)$. In fact, no UMP test exists in $\mathcal{D}^\alpha(h)$ because an MP test for a fixed $\mu = \mu_1 > 0$ in general depends on both q and μ_1. However, the above result implies (f) as demonstrated below.

Let

$$K_i^\alpha = \bigcap_{h \in F_i(0)} K^\alpha(h), \quad i = 0, 1. \tag{2.6}$$

LEMMA 2.1. (1) $\mathcal{D}_0^\alpha = K_0^\alpha$. (2) $\mathcal{D}_0^\alpha = K_1^\alpha$.

Proof. If $\phi \in K_0^\alpha$, then $\phi \in K^\alpha(h)$ for all $h \in F_0(0)$ which implies $E_0^v[\phi(v,w)] \leq \alpha$, a.a.(w.h) for any $h \in F_0(0)$. This immediately gives $E[\phi(v,w)|h] \leq \alpha$ for all $h \in F_0(0)$ so that $\phi \in \mathcal{D}_0^\alpha$. To show the converse, suppose $\phi \in \mathcal{D}_0^\alpha$ and there exists $h_1 \in F_0(0)$ such that h_1 gives positive mass to the set $S = \{w > 0 | E_0^v[\phi(v,w)] > \alpha\}$. Clearly S is independent of any $h \in F_0(0)$ since, by Proposition 1, the density of v under $\mu = 0$ is $r_0(v)$, independent of h. Denote by $r_1(w)$ the marginal density of w under h_1, which is of the form given in (3) of Proposition 1. Since $r_1(w)$ is absolutely continuous with respect to Lebesgue measure, S has a positive Lebesgue measure, which in turn implies that S contains a bounded nonempty open set A. Now define a density on R^n by $h_2(\underset{\sim}{x}) = I_A(\|\underset{\sim}{x}\|^2)/\int_{R^n} I_A(\|\underset{\sim}{x}\|^2)d\underset{\sim}{x}$ where $I_A(\cdot)$ denotes the indicator function of the set A. Obviously, $h_2 \in F_0(0)$ and h_2 gives the whole mass to the set A. Therefore, $E[\phi|h_2] = E\{E_0^V[\phi(v,w)]|r_2\} > \alpha$ where $r_2(w)$ is the density of w under h_2. This implies $\phi \notin \mathcal{D}_0^\alpha$, a contradiction. This proves (1).

To prove (2), note that $\mathcal{D}_0^\alpha \subset K_1^\alpha$ by (1) and the fact that $F_1(0) \subset F_0(0)$. To show the converse, suppose $\phi \in K_1^\alpha$ which means $E[\phi|h] \leq \alpha$ for all $h \in F_1(0)$. By the completeness of w for the family $\{N(0,\sigma^2 I_n)|\sigma^2 > 0\}$ whose densities are included in $F_1(0)$, we get $E_0^V[\phi(v,w)] \leq \alpha$, a.e. (Lebesgue), implying $\phi \in \mathcal{D}_0^\alpha$. This proves (2).

As a by-product, we get the following interesting facts.

<u>COROLLARY 2.1</u>. For problem (2.1), ϕ_1 is UMP in $K_1^\alpha = \mathcal{D}_0^\alpha$. For problem (2.2), ϕ_1 is UMP in $\mathcal{D}_0^\alpha$.

Next we consider problem (2.3). Note that although the alternatives for the two problems (2.2) and (2.3) are the same (K_1), the null hypotheses H_0 and H_1 are different as pointed out below.

<u>LEMMA 2.2.</u> $\mathcal{D}_0^\alpha \subsetneq \mathcal{D}_1^\alpha$.

<u>Proof</u>. Since $F_0(0) \supset F_1(0)$, it is clear that $\mathcal{D}_0^\alpha \subset \mathcal{D}_1^\alpha$. [illegible] test $\phi \in \mathcal{D}_1^\alpha$ but $\notin \mathcal{D}_0^\alpha$, let $\phi(w) = 1$ for $1 < w < 1 + a$, $(a > 0)$, and $\phi(w) = 0$ otherwise. Then for [illegible] in $F_1(0)$, from [illegible] we get

$$E_f[\phi(w)] = \int \phi(w) c_n w^{n/2-1} [illegible] (w)dw = c_n(1+a)^{n/2-1} [illegible]$$

since $q(1) \le 1$ because $q(\int \|x\|^2)dx = 1$ and q is non[illegible] for sufficiently small $a > 0$, $E_f[\phi(w)] \le \alpha$ for all $f \in$ [illegible] hand, defining $f_1(x) = I_B(\|x\|^2)/\int I_B(\|x\|^2)dx$ with B [illegible] it is clear that $f_1(x) \in F_0(0)$ but $\notin F_1(0)$, and E [illegible] the proof.

In view of the above lemma, it follows that [illegible] for problem (2.2), the same need not be true [illegible] fact, we show below that the [illegible] holds only for [illegible] we [illegible] does [illegible]

[illegible]

LEMMA [illegible]

[illegible]

[illegible]

THEOREM [illegible] Assume [illegible], and for $f_1(x) = q_1(\|x - \mu\|^2)$ in $F_1(\mu)$ where [illegible] for testing $H_1: h \in F_1(0)$ versus $h = f_1$ with $\mu \ne 0$, ϕ_1 is [illegible]

<u>Proof</u>. Fix $\mu = \mu_1 > 0$ and choose $f_1(\underset{\sim}{x}) = q_1(\|\underset{\sim}{x} - \mu_1 \underset{\sim}{1}\|^2) = f_1^{\mu_1}(\underset{\sim}{x})$ (say), $f_0(\underset{\sim}{x}) = q_1(\|\underset{\sim}{x}\|^2 + 2\beta\mu_1\sqrt{n}\|\underset{\sim}{x}\| + n\mu_1^2)/J$ in Lemma 2.3, where $0 < \beta < 1$ and $J = \int_{R^n} q_1(\|\underset{\sim}{x}\|^2 + 2\beta\mu_1\sqrt{n}\|\underset{\sim}{x}\| + n\mu_1^2)d\underset{\sim}{x} > 0$. Note that J can be written as

$$J = \int_{R^n} q_1((\|\underset{\sim}{x}\| + \beta\mu_1\sqrt{n})^2 + \delta)d\underset{\sim}{x}$$

$$= c_1\int_0^\infty q_1((r + \beta\mu_1\sqrt{n})^2 + \delta)r^{n-1}dr$$

$$= c_1\int_\tau^\infty q_1(r^2 + \delta)(r - \tau)^{n-1}dr < \infty$$

since $1 = \int_{R^n} q_1(\|\underset{\sim}{x}\|^2)dx = c_1\int_0^\infty q_1(r^2)r^{n-1}dr$. Here c_1 is a constant, $\tau = \beta\mu_1\sqrt{n} > 0$ and $\delta = n\mu_1^2(1 - \beta^2) > 0$. Clearly, $f_0(x) \in F_1(0)$ because $q_1(\cdot)$ is nonincreasing in $\|\underset{\sim}{x}\|$ for $\tau > 0$. Hence, by Lemma 2.3 with $K = J$ and $\nu = 0$, the test ψ with critical region

$$q_1(\|\underset{\sim}{x} - \mu_1\underset{\sim}{1}\|^2) \geq q_1(\|\underset{\sim}{x}\|^2 + 2\beta\mu_1\sqrt{n}\|\underset{\sim}{x}\| + n\mu_1^2) \tag{2.8}$$

is MP for testing H_1: $h \in F_1(0)$ versus $h = f_1^{\mu_1}$ provided $E_f[\psi] \leq \alpha$ for all $f \in F_1(0)$. Since q_1 is nonincreasing, (2.8) yields the test ϕ_1 with the critical region $v \geq -\beta$. However, by (3) of Proposition 1, the distribution of v when $\mu = 0$ is independent of $h \in F_0(0) \supset F_1(0)$. Hence ϕ_1 is MP for testing H_1 versus $f = f_1^{\mu_1}$ and hence UMP against f_1 because ϕ_1 is independent of the fixed $\mu_1 > 0$. Finally, note that $P\{v > -\beta\} \geq P\{v > 0\} = \frac{1}{2}$ for any $h \in F_1(0)$. Hence the theorem.

Since the UMP test ϕ_1 for a fixed q_1 in $F_1(\mu)$ is independent of q_1, we immediately have

<u>COROLLARY 2.2</u>. For problem (2.3), ϕ_1 is UMP whenever $\frac{1}{2} \leq \alpha < 1$.

3. TWO-SIDED TESTING PROBLEMS

In this section we deal with the following two testinq problems:

$$\text{For a fixed } h \in F_2(\mu), \quad \mu = 0 \text{ versus } \mu \neq 0 \text{ unknown} \tag{3.1}$$

$$H_2\colon h \in F_2(0) \text{ versus } K_2\colon h \in F_2(\mu), \quad \mu \neq 0 \text{ unknown.} \tag{3.2}$$

Note that the classes of level α test functions relevant for these two problems are respectively $C^\alpha(h)$ and C_2^α defined in (1.10) and (1.7). As in Section 2, let

$$\begin{aligned} \mathcal{D}^\alpha(h) &= \{\phi \in \mathcal{D}^\alpha \mid \phi(v,w) \in C^\alpha(h)\}, \quad h \in F_2(0) \\ \mathcal{D}_2^\alpha &= \{\phi \in \mathcal{D}^\alpha \mid \phi(v,w) \in C_2^\alpha\} \end{aligned} \tag{3.3}$$

where $\mathcal{D}^\alpha$ is the class of all level α tests based on (v,w). Further, let

$$K(v\colon \mu,w) = g(v,w;\mu)\Big/\int g(v,w\colon \mu)\,dv \tag{3.4}$$

be the conditional density of v given w where $g(v,w\colon \mu)$ is given in (2) of Proposition 1. Note that $K(v\colon o,w) = r_0(v)$. For a test function $\phi \in \mathcal{D}^\alpha$, let

$$\pi(\phi,(\mu,w,h)) \equiv E[\phi(v,w) \mid K(\cdot\colon \mu,w)] = \int_{-1}^{1} \phi(v,w)K(v\colon \mu,w)\,dv \tag{3.5}$$

be the conditional power of ϕ, given w, which reduces to the conditional size

$$\pi(\phi,(o,w,h)) \equiv E_0^v[\phi(v,w)] = \int_{-1}^{1} \phi(v,w)r_0(v)\,dv \tag{3.6}$$

under $\mu = 0$. Finally, let $KU^\alpha(h)$ and $KS^\alpha(h)$ be respectively the class of conditional unbiased tests of level α in $\mathcal{D}^\alpha$ and the class of conditional similar tests of level α in $\mathcal{D}^\alpha$ for $h \in F_2(\mu)$ defined as

$$KU^{\alpha}(h) = \{\phi \in \mathcal{D}^{\alpha}(h) \mid E_0^{v}[\phi(v,w)] \leq \alpha \quad \text{a.a.}(w,h), \tag{3.7}$$
$$\pi(\phi,(\mu,w,h)) \geq \alpha \text{ for all } \mu \neq 0 \quad \text{a.a.}(w,h)\}$$
$$KS^{\alpha}(h) = \{\phi \in \mathcal{D}_2^{\alpha} \mid E_0^{v}[\phi(v,w)] = \alpha \quad \text{a.a.}(w.h)\}.$$

It is easy to verify that for each fixed $h \in F_2(\mu)$, the conditional power $\pi(\phi,(\mu,w,h))$ is continuous a.a.(w,h) at $\mu = 0$, which yields the relation

$$KU^{\alpha}(h) \subset KS^{\alpha}(h) \text{ for each fixed } h \in F_2(\mu). \tag{3.8}$$

We are now ready to state the first main result of this section.

<u>THEOREM 3.1</u>. For problem (3.1), the two-sided t-test ϕ_2 defined in (1.2) is UMP in the class of tests in $KS^{\alpha}(h)$ satisfying

$$E_0^{v}[v\phi(v,w)] = 0 \quad \text{a.a.}(w,h). \tag{3.9}$$

<u>Proof</u>. Write $h(x) = q(\|x - \mu 1\|^2)$ where q is fixed nonincreasing and convex, and consider the problem of testing $\mu = 0$ versus $\mu = \mu_1 \neq 0$ in the conditional density $k(v;\mu,w)$ in (3.4). Obviously $KS^{\alpha}(h)$ is the class of size α tests for this problem. Now by the generalized Neyman-Pearson lemma, a test ϕ^* which maximizes the conditional power $\pi(\phi,(\mu_1,w,h))$ subject to (3.9) is given by

$$\phi^*(v,w) = \begin{cases} 1 & \text{if } k(v;\mu_1,w) > c_1 r_0(v) + c_2 v r_0(v) \\ 0 & \text{otherwise} \end{cases} \tag{3.10}$$

where c_1 and c_2 are constants chosen to satisfy the size α condition $E_0^{v}[\phi^*] = \alpha$ and (3.9). Using (2) of Proposition 1 and the convexity of q, it follows easily that ϕ^* is equivalent to

$$\phi^*(v,w) = \begin{cases} 1 & \text{if } v > b \quad \text{or} \quad v < a \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

which is clearly independent of μ_1. From (3) of Proposition 1, due to the symmetry of $r_0(v)$, we get that the constants a and b satisfying $E_0^v(\phi^*) = \alpha$ and (3.9) must obey the relation $-a = b = c$ (say). Therefore $\phi^* = \phi_2$, and hence for any $\phi \in KS^\alpha(h)$ satisfying (3.9) and for any $\mu \neq 0$, we get

$$\pi(\phi_2,(\mu,w,h)) \geq \pi(\phi,(\mu,w,h)) \quad \text{a.a.}(w,h). \tag{3.12}$$

Taking expectations with respect to w in both sides of (3.12) proves the theorem.

In view of the relation in (3.8) and the fact that $\phi_2 \in KU^\alpha(h)$, we immediately conclude

<u>COROLLARY 3.1</u>. For problem (3.1), ϕ_2 is UMP in the class of tests in $KU^\alpha(h)$ satisfying (3.9).

We next consider problem (3.2). Let

$$KU^\alpha = \bigcap_{h \in F_2(0)} KU^\alpha(h), \quad KS^\alpha = \bigcap_{h \in F_2(0)} KS^\alpha(h) \tag{3.13}$$

denote respectively the class of conditional unbiased tests of level α in $\mathcal{D}^\alpha$ and the class of conditional similar tests of level α in $\mathcal{D}^\alpha$ appropriate for this problem. Also let $\mathcal{U}^\alpha$ and $\mathcal{S}^\alpha$ be respectively the class of unbiased tests of level α in $\mathcal{D}_2^\alpha$ and the class of similar tests of level α in $\mathcal{D}_2^\alpha$ defined as

$$\begin{aligned} \mathcal{U}^\alpha &= \{\phi \in \mathcal{D}_2^\alpha \mid \pi(\phi,(0,h)) \leq \alpha \text{ for any } h \in F_2(0) \\ &\qquad \pi(\phi,(\mu,h)) \geq \alpha \text{ for any } h \in F_2(\mu) \text{ and } \mu \neq 0\} \\ \mathcal{S}^\alpha &= \{\phi \in \mathcal{D}_2^\alpha \mid \pi(\phi,(0,h)) = \alpha \text{ for any } h \in F_2(0)\}. \end{aligned} \tag{3.14}$$

where $\pi(\phi,(\mu,h)) \equiv E[\phi(v,w)]$ is the power function of ϕ under $h \in F_2(\mu)$.

The various relationships amongst these four classes are given below.

LEMMA 3.1. (1) $\mathcal{KU}^\alpha \subset \mathcal{U}^\alpha$

(2) $\mathcal{U}^\alpha \subset \mathcal{S}^\alpha$

(3) $\mathcal{KS}^\alpha = \mathcal{S}^\alpha$.

Proof. (1) follows from the definitions of $\mathcal{KU}^\alpha$ and $\mathcal{U}^\alpha$. (2) follows from the easily verifiable fact that for any $\phi \in \mathcal{D}_2^\alpha$ and any $h \in F_2(\mu)$, $\pi(\phi,(\mu,h))$ is continuous in μ. To show (3), note that $\mathcal{KS}^\alpha \subset \mathcal{S}^\alpha$ follows from the definitions. On the other hand, since $N(\mu) \equiv \{N(\mu,\sigma^2) | \sigma^2 > 0\} \subset F_2(\mu)$ and w is complete for $N(0) \subset F_2(0)$, it follows that $\pi(\phi,(0,h)) = E_0^w[E_0^v\{\phi(v,w)|w\}] = \alpha$ necessarily implies

$$E_0^v[\phi(v,w)] = \alpha \quad \text{a.a.}(w,h) \quad \text{for all } h \in F_2(0),$$

which shows $\mathcal{S}^\alpha \subset \mathcal{KS}^\alpha$. This completes the proof of the lemma.

Our main result for problem (3.2) is the following.

THEOREM 3.2. For problem (3.2), the two-sided t-test ϕ_2 is UMP in $\mathcal{S}^\alpha$ i.e., UMP similar.

Proof. By Theorem 3.1 and (3) of Lemma 3.1, it is enough to show that any $\phi \in \mathcal{S}^\alpha$ satisfies (3.9). Assuming $h \in F_2(\mu)$ is the density of $N(\mu \underset{\sim}{1},\sigma^2 I_n)$, it follows that when $\mu = 0$,

$$E_0^v[v\phi(v,w)] = \alpha E_0^v[v] = 0 \tag{3.15}$$

because the derivative of $\pi(\phi,(\mu,h))$ at $\mu = 0$ is zero and because w is complete for $N(0) \equiv \{N(\underset{\sim}{0},\sigma^2 I_n) | \sigma^2 > 0\}$. However, by Proposition 1, (3.15) is true for any $h \in F_2(0)$. This completes the proof.

Since $\phi_2 \in \mathcal{U}^\alpha$, by (2) of Lemma 3.1 we immediately get

COROLLARY 3.2. For problem (3.2), ϕ_2 is UMP in $\mathcal{U}^\alpha$, i.e. UMP unbiased.

4. t-TEST WITH INVARIANCE

It is well-known (Kariya (1981)) that when the density of $\underset{\sim}{x}$ is given by (1.3), the one-sided t-test ϕ_1 is UMPI for testing $\mu = 0$ versus $\mu > 0$ for any fixed nonincreasing q and the two-sided t-test ϕ_2 is UMPI for testing $\mu = 0$ versus $\mu \neq 0$ for any fixed nonincreasing and convex q. The groups leaving the problems invariant are respectively given by $G_1 \equiv R_+ \times \tilde{O}(n)$ and $G_2 \equiv R_* \times \tilde{O}(n)$, where $R_+ = \{a > 0\}$, $R_* = \{a \in R | a \neq 0\}$, $\tilde{O}(n) = \{\Gamma \in O(n) | \Gamma \underset{\sim}{1} = \underset{\sim}{1}\}$, and $O(n)$ is the group of n x n orthogonal matrices. Maximal invariants under G_1 and G_2 with group actions $(\gamma,\Gamma)(x) = \gamma\Gamma\underset{\sim}{x}, (\gamma,\Gamma) \in G_i$, $i = 1, 2$, are respectively v and $|v|$, and the classes of invariant level α tests are respectively given by

$$\mathcal{J}_1^\alpha = \{\phi \in \mathcal{D}^\alpha | \phi \text{ is based on v only, } E_0^v[\phi] \leq \alpha\} \tag{4.1}$$

$$\mathcal{J}_2^\alpha = \{\phi \in \mathcal{J}_1^\alpha | \phi(-v) = \phi(v)\}. \tag{4.2}$$

Since the null distribution of v does not depend on q (vide Kariya (1981)), it follows from (2.4) that

$$\mathcal{J}_2^\alpha \subset \mathcal{J}_1^\alpha \subset \mathcal{D}_0^\alpha \subset \mathcal{D}_1^\alpha \subset \mathcal{D}^\alpha(h), \quad h \in F_2(0). \tag{4.3}$$

In fact, the following result holds.

LEMMA 4.1. $\mathcal{J}_1^\alpha \subsetneq \mathcal{D}_0^\alpha$.

<u>Proof</u>. Take any test function $\psi(w) \neq$ constant and any $\phi(v) \in \mathcal{J}_1^\alpha$. Then the test defined by $\psi(w)\phi(v) \in \mathcal{D}_0^\alpha$ because $E_0^v[\psi(w)\phi(v)] = \psi(w)E_0^v[\phi(v)] \leq \alpha$. However, $\psi(w)\phi(v) \notin \mathcal{J}_1^\alpha$, completing the proof.

Consequently, for the one-sided testing problem, the results in Corollary 2.1 are somewhat stronger than the result that ϕ_1 is UMPI.

An analogous result is obtained below for the two-sided testing problem. Recall the definitions of S^α and $\mathcal{J}_2^\alpha$ given in (3.14) and (4.2) respectively.

<u>LEMMA 4.2</u>. $\mathcal{J}_2^\alpha \not\supseteq S^\alpha$.

<u>Proof</u>.# Note from Proposition 1 that $h \in F_2(0)$, v and w are independent and the densities are respectively given by $r_0(v)$ and $r_1(w)$ where q depends on h. The conditional size condition $E_0^v[\phi(v,w)] = \alpha$ and the conditional similarity condition (3.9) can be expressed as

$$\int \phi(v,w)r_0(v)dv = \alpha \qquad \text{a.a.}(w,h \in F_2(0)) \tag{4.4}$$

$$\int v\phi(v,w)r_0(v)dv = 0 \qquad \text{a.a.}(w,h \in F_2(0)). \tag{4.5}$$

Now take two tests $\phi_1(v)$ and $\phi_2(v)$ based on v only which are functionally independent and satisfy (4.5) with

$$\beta_i = \int \phi_i(v)r_0(v)dv, \quad i = 1, 2$$

and β_1 [illegible] α [illegible] β_2 [illegible] 0. Next choose any two tests $\psi_1(w)$ and $\psi_2(w)$ based on w only which satisfy

$$\beta_1\psi_1(w) + \beta_2\psi_2(w) \; [illegible] \; \alpha.$$

Finally, define the test $\phi^*(v,w)$ [illegible] $\phi_1(v)\psi_1(w) + \phi_2(v)\psi_2(w)$. Clearly ϕ^* satisfies $E_0^v[\phi^*(v,w)]$ [illegible] α and $\phi^*(\neq$ constant) does depend nontrivially on w. This completes the proof

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Clearly $\mathcal{J}_2^\alpha \subset S^\alpha$. We will construct a test $\phi^*(v,w) \in S^\alpha$ but $\notin \mathcal{J}_2^\alpha$.

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